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## REDUCED BASIS TECHNIQUE FOR EVALUATING THE 02 7556 SENSITIVITY OF THE NONLINEAR VIBRATIONAL RESPONSE OF COMPOSITE PLATES

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Abstract—A reduced basis technique and a computational procedure are presented for generating the nonlinear vibrational response, and evaluating the first-order sensitivity coefficients of composite plates (derivatives of the nonlinear frequency with respect to material and geometric parameters of the plate). The analytical formulation is based on a form of the geometrically nonlinear shallow shell theory with the effects of transverse shear deformation, rotatory inertia and anisotropic material behavior included. The plate is discretized by using mixed finite element models with the fundamental unknowns consisting of both the nodal displacements and the stress-resultant parameters of the plate. The computational procedure can be conveniently divided into three distinct steps. The first step involves the generation of various-order perturbation vectors, and their derivatives with respect to the material and lamination parameters of the plate, using Linstedt-Poincaré perturbation technique. The second step consists of using the perturbation vectors as basis vectors, computing the amplitudes of these vectors and the nonlinear frequency of vibration, via a direct variational procedure. The third step consists of using the perturbation vectors, and their derivatives, as basis vectors and computing the sensitivity coefficients of the nonlinear frequency via a second application of the direct variational procedure. The effectiveness of the proposed technique is demonstrated by means of numerical examples of composite plates.

### NOTATION

	HOTATION
$E_L, E_T$	elastic moduli of the individual layers in the
	direction of fibers and normalto it, respect-
	ively
[F]	global flexibility matrix of the plate model
$F_{ijk}$ and $G_{ijks}$	nonlinear arrays in the reduced equations
$G_{LT}^{\gamma \wedge}, G_{TT}^{-ij \wedge 3}$	shear moduli in the plane of fibers and
-11, -11	normal to it, respectively
$\{G(X)\}$	vector of nonlinear terms
$\{H\}$	· · · · · · · · · · · · · · · · · · ·
firt	vector of stress-resultant parameters of the
,	plate model
h	total thickness of the plate
$K_{ij}$	linear stiffness arrays in the reduced
	equations
[M]	consistent mass matrix of the plate model
$M_{ij}$	mass coefficients in the reduced equations
m	Fourier harmonic (wave number)
n	highest-order term retained in the pertur-
	bation series
r	total number of approximation vectors
[S]	strain-displacement matrix
T	kinetic energy of the plate
t	time
$u_1, u_2, w$	displacement components of the reference
1727	(middle surface) in the coordinate directions
{ <i>X</i> }	vector of nodal displacements of the plate
(** )	model
v v v	
$x_1, x_2, x_3$	Cartesian coordinate system
€	perturbation parameter
$\phi_1,\phi_2$	rotation components of the reference
,	(middle) surface of the plate
$\psi_i$	amplitudes of approximation vectors
Ω	$=\omega^2$
ω	circular frequency of vibration

λ	cretized plate material or lamination parameter of the plate
Range of in	ndices
i	zero to highest-order exponent of $\epsilon$ in the perturbation expansion of $\omega^2$
i, j, k, s	1 to <i>r</i>
m	zero to number of harmonics used in the temporal approximation
Subscript	

major Poisson's ratio of the individual layers Hellinger-Reissner functional of the dis-

### INTRODUCTION

center

transpose

Superscript

Significant advances have been made in the development of effective analytical and numerical techniques for the nonlinear vibration analysis of composite plates (see, for example, [1–9]). Reviews of some of these techniques are contained in survey papers [10–13] and two monographs [14, 15]. However, the use of nonlinear vibration analysis in automated optimum design of composite plates requires the availability of efficient techniques for calculating the sensitivity of the nonlinear vibration response to variations in the design

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variables. The sensitivity coefficients (derivatives of the nonlinear vibration frequencies with respect to design variables) can be used to: (a) determine a search direction in the direct application of nonlinear mathematical programming algorithms. When approximation concepts are used for optimum design of beams, sensitivity coefficients are used to construct explicit approximations for the critical, and potentially critical, behavior constraints; (b) generate an approximation for the nonlinear frequency of a modified beam (in conjunction with a reanalysis technique); (c) assess the effects of uncertainties. in the material and geometric parameters of the beam, on the nonlinear frequency; and, (d) predict the changes in the nonlinear frequency due to changes in the beam parameters. Two general procedures are currently being used for calculating the sensitivity coefficients of nonlinear structural response. The two approaches are (see, for example, [16-19]): the direct differentiation method and adjoint variable method. The first procedure is based on the implicit differentiation of the equations that describe the nonlinear response with respect to the desired parameters and the solution of the resulting sensitivity equations. In the adjoint variable method an adjoint physical system is introduced whose solution permits rapid evaluation of the desired sensitivity coefficients. Both procedures can be applied to either the governing discrete equations or to the functional of the variational formulation of the structure (with a consequent change in the order of discretization and implicit differentiation).

Recently, the reduced basis technique, which was first presented in [20], was adapted to the nonlinear

vibration analysis of composite plates in [21]. The present paper extends the reduced basis technique to the evaluation of the sensitivity coefficients. A geometrically nonlinear shallow shell theory is used, with the effects of transverse shear deformation, rotatory inertia and anisotropic material behavior included. The perturbation vectors are used as coordinate functions in evaluating the nonlinear frequencies. A combination of the perturbation vectors and their derivatives with respect to the material and lamination parameters of the panel is used, in conjunction with a direct differentiation approach, for approximating the sensitivity coefficients.

### MATHEMATICAL FORMULATION

### Variational equations

The spatial discretization is performed by using two-field mixed finite element model with the fundamental unknowns consisting of nodal displacements and stress-resultant parameters. For undamped free vibrations of the discretized plate, the variational equations used in evaluating the nonlinear vibrational response and the sensitivity coefficients can be written in the following form

$$\delta \int_{t_1}^{t_2} (T - \pi) \, \mathrm{d}t = 0 \tag{1}$$

and

$$\delta \int_{t_1}^{t_2} \left( \frac{\partial T}{\partial \lambda} - \frac{\partial \pi}{\partial \lambda} \right) dt = 0$$
 (2)

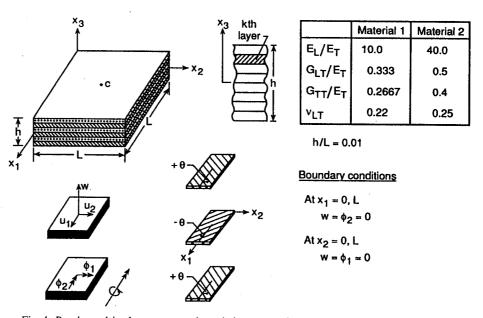


Fig. 1. Panels used in the present study and sign convention for displacements and rotations.

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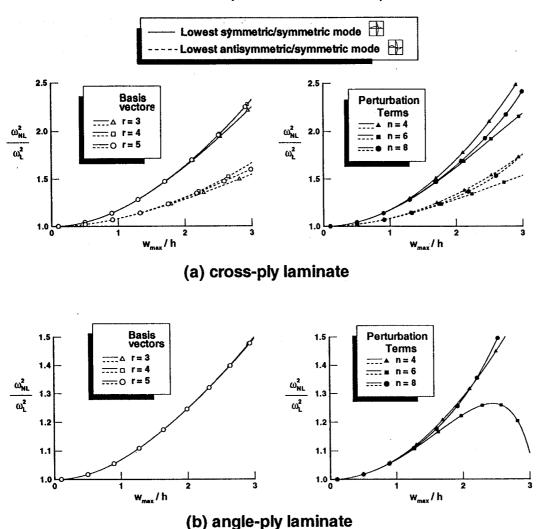


Fig. 2. Convergence of nonlinear frequencies  $\omega_{NL}^2/\omega_L^2$  obtained by the reduced basis and perturbation techniques. Two-layered cross-ply and angle-ply square plates (see Fig. 1). For the antisymmetric/symmetric mode  $w_{\text{max}}$  occurs at  $x_1 = 0.25L$ ,  $x_2 = 0.5L$ .

subject to the condition that at times  $t=t_1$  and  $t=t_2$ , the first variations of the fundamental unknowns (generalized nodal displacements and stress-resultant parameters) and their sensitivity coefficients (derivatives with respect to  $\lambda$ ) vanish. In eqns (1) and (2),  $\lambda$  refers to a typical material, lamination or geometric parameter of the plate; T and  $\pi$  are the kinetic energy and Hellinger-Reissner functional of the discretized plate given by

$$T = \frac{1}{2} \left\{ \frac{\partial X}{\partial t} \right\}^{\mathrm{T}} [M] \left\{ \frac{\partial X}{\partial t} \right\}$$
 (3)

$$\pi = \{H\}^{\mathsf{T}} \left(\frac{-1}{2} [F] \{H\} + [S] \{X\} + \{G(X)\}\right),$$

where  $\{H\}$  is the vector of stress-resultant parameters;  $\{X\}$  is the vector of the nodal dis-

placements; [M] is the consistent mass matrix; [F] is the global flexibility matrix; [S] is the strain-displacement matrix;  $\{G(X)\}$  is the vector of non-linear terms; and superscript T denotes transposition.

The application of the reduced basis technique to the nonlinear vibration and sensitivity analyses of the plate can be conveniently divided into three distinct steps: (1) generating perturbation vectors, and their derivatives with respect to the material, lamination and geometric parameters of the plate, using the Linstedt-Poincaré perturbation technique; (2) using the perturbation vectors as basis (or global approximation) vectors, and computing the amplitudes of these vectors and the nonlinear vibration frequency via a direct variational technique, in conjunction with the method of harmonic balance; and (3) using the perturbation vectors and their derivatives as basis vectors and

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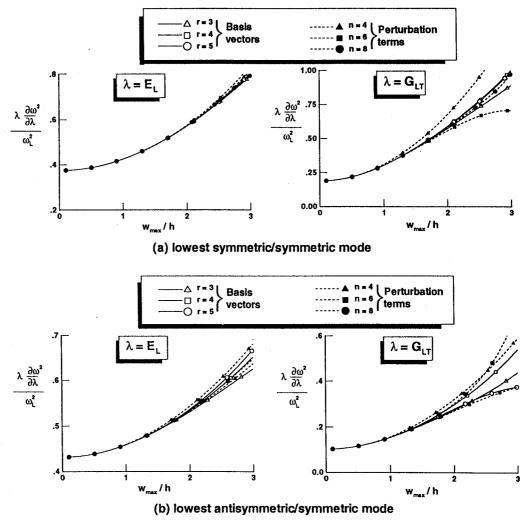


Fig. 3. Convergence of the normalized sensitivity coefficients  $\lambda (\partial \omega_{NL}^2/\partial \lambda)/\omega_L^2$  for the lowest symmetric/symmetric and antisymmetric/symmetric vibration modes, obtained by the reduced basis and perturbation techniques. Two-layered cross-ply square plates (see Fig. 1). For the antisymmetric/symmetric mode  $w_{\text{max}}$  occurs at  $x_1 = 0.25L$ ,  $x_2 = 0.5L$ .

evaluating the sensitivity coefficients of the nonlinear frequency via a second application of the direct variational technique. The procedure is described subsequently.

Generation of perturbation vectors and their derivatives

For the purpose of generating basis vectors, a new independent variable  $\tau = \omega t$  is introduced, where  $\omega$  is the nonlinear circular frequency. The following expansion is used for  $\Omega = \omega^2$ , in terms of a small parameter  $\epsilon$ 

$$\omega^2 = \Omega(\epsilon) = \sum_{i=0} \Omega^{(i)} \epsilon^i.$$
 (5)

retained in the expansion. The vectors  $\{H\}$  and  $\{X\}$  and  $\omega^2$ , eqns (5) and (7), into eqns (3), (4) and

 $\{X\}$  are also expanded in perturbation series of the form

$$\left\{ \begin{array}{l} H(t,\epsilon) \\ X(t,\epsilon) \end{array} \right\} = \sum_{i=1}^{\infty} \left\{ \begin{array}{l} H(\tau) \\ X(\tau) \end{array} \right\}^{(i)} \epsilon^{i}.$$
 (6)

Each of the time-dependent vectors,  $\{H(\tau)\}^{(i)}$  and  $\{X(\tau)\}^{(i)}$ , are expanded in a Fourier series in  $\tau$ , and therefore

$$\begin{cases} H(t,\epsilon) \\ X(t,\epsilon) \end{cases} = \sum_{i=1}^{\infty} \left( \sum_{m=0}^{i} \begin{cases} H \\ X \end{cases}^{(i,m)} \cos m\omega t \right) \epsilon^{i}.$$
 (7)

The equations used in generating the vectors

$$\left\{ \begin{matrix} H \\ X \end{matrix} \right\}^{(i,m)}$$

Only the even values of i (i = 0, 2, 4, ...) are are obtained by substituting the expansions for  $\{H\}$ ,

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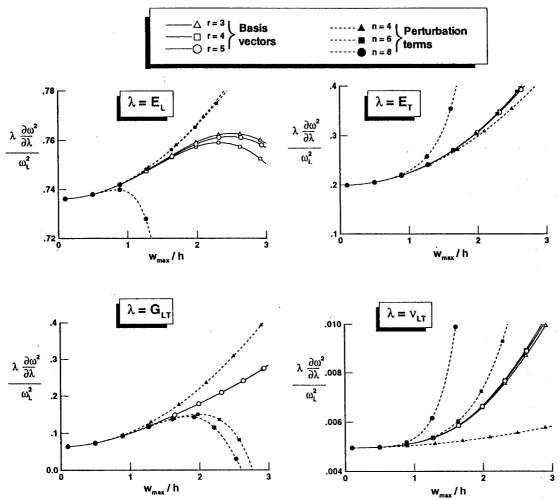


Fig. 4. Convergence of the normalized sensitivity coefficients  $\lambda (\partial \omega_{NL}^2/\partial \lambda)/\omega_L^2$  for the lowest symmetric vibration mode, obtained by the reduced basis and perturbation techniques. Two-layered angle-ply composite plates (see Fig. 1).

(1); converting each term into the first power of cosine functions in t; and setting the like terms of  $\epsilon$  and m to zero. This leads to a recursive set of linear equations in  $\{H\}^{(i,m)}$  and  $\{X\}^{(i,m)}$ . The explicit form of these equations is given in [21]. Note that the linear free vibration problem corresponds to i=m=1. For each vibration mode (i.e., a prescribed pair of eigenvalue and eigenvector) a set of vectors  $\{H\}^{(i,m)}$  and  $\{X\}^{(i,m)}$  can be generated. The multipliers of  $\epsilon^i$ , i.e., the quantity between parentheses in eqn (7), will henceforth be referred to as the perturbation vectors.

The derivatives of  $\omega^2$ ,  $\{H\}$  and  $\{X\}$  with respect to the material, lamination and geometric parameters of the plate are given by

$$\frac{\partial \omega^2}{\partial \lambda} = \frac{\partial \Omega}{\partial \lambda} = \sum_{i=0}^{\infty} \frac{\partial \Omega^{(i)}}{\partial \lambda} \epsilon^i$$
 (8)

and

$$\frac{\partial}{\partial \lambda} \left\{ \begin{matrix} H(t,\epsilon) \\ X(t,\epsilon) \end{matrix} \right\} = \sum_{i=1}^{\infty} \left( \sum_{m=0}^{i} \frac{\partial}{\partial \lambda} \left\{ \begin{matrix} H \\ X \end{matrix} \right\}^{(i,m)} \cos m\omega t \right) \epsilon^{i}.$$
(9)

The equations used in generating

$$\left\{\frac{\partial H}{\partial \lambda}\right\}^{(i,m)}, \quad \left\{\frac{\partial X}{\partial \lambda}\right\}^{(i,m)}$$

are obtained by either: (a) differentiating the governing recursive equations in  $\{H\}^{(i,m)}$ ,  $\{X\}^{(i,m)}$  with respect to  $\lambda$ ; or (b) following the same steps used in generating the governing equations for  $\{H\}^{(i,m)}$ ,  $\{X\}^{(i,m)}$ , but using the functionals in eqn (2) of the functionals in eqn (1).

In the present study, computerized symbolic manipulation was used in generating the perturbation vectors and their derivatives with respect to  $\lambda$ . The nonzero terms are associated with either (i, m) even,

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(6)

and and

(7)

 $\{H\}$ , and

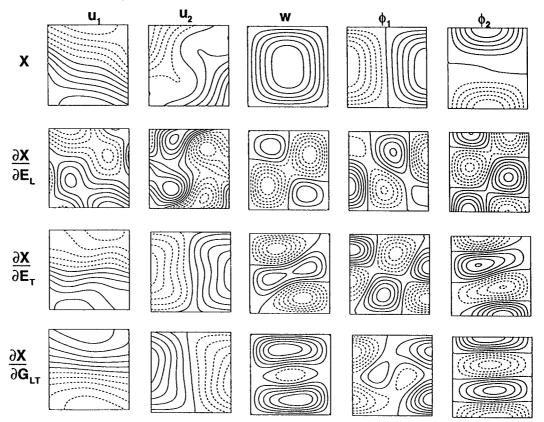


Fig. 5. Normalized contour plots of the generalized displacements and their sensitivity coefficients associated with the lowest vibration mode. Two-layered angle-ply composite plates (see Fig. 1). Spacing of the contour plots is 0.2 and the dashed lines refer to negative contours.

or (i, m) odd, and are listed in Table 1. All the nonzero coefficients correspond to even values of i + m.

Computation of amplitudes of basis vectors and nonlinear frequency

The perturbation vectors in eqn (7) are now chosen as basis vectors, and the response vectors are expressed as linear combinations of these vectors as follows:

$$\begin{Bmatrix} H \\ X \end{Bmatrix} = \sum_{i=1}^{r} \left( \sum_{m=0}^{i} \begin{Bmatrix} H \\ X \end{Bmatrix}^{(i,m)} \cos m\omega t \right) \psi_{i}, \quad (10)$$

where  $\psi_i$  are unknown parameters representing the amplitudes of the basis vectors; and r is the total number of basis vectors used. Equation (10) is substituted into eqns (1), (3) and (4). A direct variational technique is used in conjunction with the method of harmonic balance to approximate eqn (1) by a system of nonlinear algebraic equations in  $\psi_i$  (i = 1 to r) and  $\omega^2$ . The additional equation needed to solve the system is obtained by prescribing either one of the displacement components (linear combination of  $\psi_i$ ) or one of the parameters of  $\psi_i$ . The additional equation will henceforth be referred to

as the constraint condition. The form of the nonlinear algebraic equations in  $\psi_i$  and  $\omega^2$  is given in Appendix I.

Evaluation of the sensitivity coefficients of the nonlinear vibrational response

The perturbation vectors and their derivatives with respect to  $\lambda$  are used in approximating the derivatives of the response vectors as follows:

$$\frac{\partial}{\partial \lambda} \begin{Bmatrix} H \\ X \end{Bmatrix} = \sum_{i=1}^{r} \left( \sum_{m=0}^{i} \frac{\partial}{\partial \lambda} \begin{Bmatrix} H \\ X \end{Bmatrix}^{(i,m)} \cos m\omega t \right) \psi_{i}$$

$$+ \sum_{i=1}^{r} \left( \sum_{m=0}^{i} \begin{Bmatrix} H \\ X \end{Bmatrix}^{(i,m)} \cos m\omega t \right) \bar{\psi}_{i}, \quad (11)$$

Table 1. Pairs of (i, m) for which the perturbation vectors are nonzero

i/m	0	1	2	3	4	5	6
ī		1,1					
2	2,0	•	2,2				
3		3,1	•	3,3			
4	4,0	•	4,2	- ,	4,4		
5		5,1	,-	5,3	.,.	5,5	
6	6,0	,	6,2	. ,-	6,4	- ,0	6,6

where  $\bar{\psi}_i$  are unknown parameters. Equation (11) is used in conjunction with a direct variational technique and the method of harmonic balance to approximate eqn (2) by a system of linear algebraic equations in  $\bar{\psi}_i$  (i=1 to r) and  $\partial \omega^2/\partial \lambda$ . The additional equation is obtained by differentiating the constraint condition, used in evaluating  $\omega^2$ , with respect to  $\lambda$ . The form of the linear algebraic equations in  $\partial \psi_i/\partial \lambda$  and  $\partial \omega^2/\partial \lambda$  is given in Appendix II.

### NUMERICAL STUDIES

To assess the effectiveness of the proposed reduced basis technique, a number of nonlinear vibration problems of laminated composite plates have been solved by this technique. For each problem the convergence of the sensitivity coefficients obtained by the proposed technique was compared with those obtained by the perturbation technique. Herein, results are presented for typical two-layer cross-ply and angle-ply square plates (see Fig. 1). The same problems were used in [21] to demonstrate the effectiveness of the reduced basis technique for nonlinear vibration problems.

The plates were discretized by using mixed finite element models with bicubic interpolation functions for each of the generalized displacements and stress resultants. The characteristics of the finite element model are given in [22]. Because of symmetry, only one quarter of the cross-ply plate and one-half of the angle-ply plate were analyzed, and the appropriate symmetry/antisymmetry conditions were applied (see [23]). An  $8 \times 8$  grid was used for the cross-ply plate, and an  $8 \times 16$  grid was used for the angle-ply plate.

The perturbation parameter  $\epsilon$  was selected to be the coefficient of the linear vibration mode  $[\psi_1]$  in eqn (10)], and the Linstedt-Poincaré method was used to generate perturbation vectors, and their derivatives, up to order 10 of  $\epsilon$  [n=10]—where n is the highest-order term in the perturbation series of  $\Omega$ ,  $\{H\}$  and  $\{X\}$  in eqns (5), (6), (8) and (9)].

The perturbation vectors were used as coordinate vectors, and a direct variational technique was applied to determine the nonlinear frequency and the amplitudes of the coordinate vectors  $[\psi_i]$  in eqn (10)]. The perturbation vectors and their derivatives, eqn (11), were then used in conjunction with a direct variational technique to determine the parameters  $\psi_i$  and the sensitivity coefficients  $\partial \omega^2/\partial \lambda$ [eqns (11) and (B1)—Appendix II] corresponding to different values of  $\epsilon$ . The sensitivity coefficients were validated by comparing them with the finite difference results. Very close agreement was observed between both. Typical results are shown in Figs 2 and 3 for the cross-ply plate and in Figs 2, 4 and 5 for the angle-ply plate, and are discussed subsequently. Figure 2 is reproduced from [21].

The accuracy and convergence of the nonlinear vibration frequencies associated with the first symmetric/symmetric and first antisymmetric/symmetric modes for cross-ply panels, obtained by the reduced basis and perturbation techniques, are shown in Fig. 2. The corresponding convergence plots for the nonlinear vibration frequency associated with the lowest symmetric mode of the angle-ply plate are also shown in Fig. 2.

The convergence of the sensitivity coefficients  $\partial \omega^2/\partial \lambda$ , are shown in Fig. 3 for the lowest two frequencies of the cross-ply plate, and in Fig. 4 for the lowest frequency of the angle-ply plate. Each sensitivity coefficient is normalized by multiplying by  $\lambda$  and dividing by the square of the linear frequency  $\omega_L^2$ . In Figs 3 and 4, n refers to the highest-order term in the perturbation series, eqns (8) and (9), and r refers to the total number of coordinate functions [number of  $\psi_i$  in eqn (11)]. In Fig. 3,  $\lambda$  was chosen to be  $E_L$  and  $G_{LT}$ ; and in Fig. 4,  $\lambda$  was chosen to be  $E_L$ ,  $E_T$ ,  $G_{LT}$  and  $v_{LT}$ .

As expected, the accuracy of  $\partial \omega^2/\partial \lambda$  obtained by the perturbation technique deteriorates rapidly with the increase of  $w_{\rm max}/h$ , particularly for  $w_{\rm max}/h>1$  in the angle-ply plate. For both plates when  $w_{\rm max}/h\geq 2$ , the sensitivity coefficients  $\partial \omega^2/\partial \lambda$  were considerably in error. On the other hand, the convergence of the sensitivity coefficients obtained by using the reduced basis technique was reasonably fast, even for  $w_{\rm max}/h$  outside the radius of convergence of the perturbation technique. This is particularly true, up to  $w_{\rm max}/h=3.0$ , for the sensitivity coefficients with respect to  $E_T$ ,  $G_{LT}$  and  $v_{LT}$  for the angle-ply plate.

In Fig. 5 normalized contour plots are presented for the displacements  $u_1$ ,  $u_2$ , w,  $\phi_1$  and  $\phi_2$  associated with the nonlinear vibration mode at  $w_c/h = 2.0$  for the angle-ply plate. Each contour plot is normalized by dividing by its maximum absolute value. Also shown in Fig. 5 are the normalized contour plots for the sensitivity coefficients with respect to  $E_L$ ,  $E_T$  and  $G_{LT}$  at the same value of  $w_c/h$ . Note that the contour plots for the sensitivity coefficients are quite different from those of the vibration mode.

### CONCLUDING REMARKS

A reduced basis technique and a computational procedure are presented for generating the nonlinear vibrational response, and evaluating the first-order sensitivity coefficients of composite plates (derivatives of the nonlinear frequency with respect to material and geometric parameters of the plate). The analytical formulation is based on a form of the geometrically nonlinear shallow shell theory with the effects of transverse shear deformation, rotatory inertia and anisotropic material behavior included. The plate is discretized by using mixed finite element models with the fundamental unknowns consisting of both the nodal displacements and the stress-resultant parameters of the plate.

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The computational procedure can be conveniently divided into three distinct steps. The first step involves the generation of various-order perturbation vectors, and their derivatives with respect to the material and lamination parameters of the plate, using Linstedt-Poincaré perturbation technique. The second step consists of using the perturbation vectors as basis vectors, computing the amplitudes of these vectors and the nonlinear frequency of vibration, via a direct variational approach. The third step consists of using the perturbation vectors, and their derivatives, as basis vectors and computing the sensitivity coefficients of the nonlinear frequency via a second application of the direct variational procedure. The effectiveness of the reduced basis technique is demonstrated by means of numerical examples of two-layered cross-ply and angle-ply plates. The convergence of the sensitivity coefficients obtained by the reduced basis technique was compared with that of the perturbation technique.

On the basis of the present study, the following observations can be made. The reduced basis technique can be thought of as either a generalized perturbation technique in which the response vectors contain free parameters rather than fixed coefficients and the perturbation parameter need not be small; or an extension of the direct variational technique with the coordinate vectors generated by using a perturbation technique rather than chosen a priori. The successive application of the perturbation technique and the direct variational procedure, which forms the basis of the foregoing reduced basis technique, results in enhancing the effectiveness of the direct variational technique by removing (or reducing) the arbitrariness in the selection of the coordinate vectors; and extending the range of applicability of the regular perturbation technique by removing the restriction of a small perturbation parameter.

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# APPENDIX I—FORM OF THE NONLINEAR EQUATIONS IN $\psi_i$ AND $\omega$

The nonlinear algebraic equations in the amplitudes c coordinate functions  $\psi_i$ , and the frequency  $\omega$ , can be writte in the following compact form:

$$K_{ij}\psi_{i} + F_{ijk}\psi_{j}\psi_{k} + G_{ijks}\psi_{j}\psi_{k}\psi_{s} - \omega^{2}M_{ij}\psi_{i} = 0,$$
 (A1)

where the range of i, j, k, s is 1 to r; and a repeated inde in the same term denotes summation over its full range. Th arrays  $K_{ij}$ ,  $F_{ijk}$ ,  $G_{ijk}$ , and  $M_{ij}$  are obtained by using eqns (10) (3), (4) and (1); applying the method of harmonic balanc and performing the temporal integration.

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APPENDIX II—FORM OF THE LINEAR EQUATIONS IN  $\psi_i$  AND  $\partial \omega^2/\partial \lambda$ 

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The linear algebraic equations in the amplitudes of the coordinate functions  $\Psi_i$ , used in approximating the derivatives of the displacement parameters, and the sensitivity coefficients  $\partial \omega^2/\partial \lambda$ , can be written in the following compact form:

$$[(K_{ij}+2F_{ijk}\psi_k+3G_{ijks}\psi_k\psi_s)-\omega^2M_{ij}]\bar{\psi_j}-M_{ij}\psi_j\frac{\partial\omega^2}{\partial\lambda}$$

$$= - \left[ \left( \frac{\partial K_{ij}}{\partial \lambda} + \frac{\partial F_{ijk}}{\partial \lambda} \psi_k + \frac{\partial G_{ijks}}{\partial \lambda} \psi_k \psi_s \right) - \omega^2 \frac{\partial M_{ij}}{\partial \lambda} \right] \psi_j. \quad (B1)$$

Note that the  $\psi$ s appearing on the left-hand side of eqn (B1) are obtained from eqn (A1). The K, F, G and M arrays appearing on the left-hand-side of eqn (B1) are in terms of  $\{H\}^{(l,m)}$  and  $\{X\}^{(l,m)}$ . The corresponding arrays on the right-hand-side are in terms of the corresponding  $\partial/\partial\lambda$   $\{H\}^{(l,m)}$  and  $\partial/\partial\lambda$   $\{X\}^{(l,m)}$ .

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